THE LOEWY LENGTH OF THE DESCENT ALGEBRA OF D_{2m+1}.

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Abstract. In this article the Loewy length of the descent algebra of D_{2m+1} is shown to be $m+2$, for $m \geq 2$, by providing an upper bound that agrees with the lower bound in [Bonnafé and Pfeiffer, 2006].

The bound is obtained by showing that the length of the longest path in the quiver of the descent algebra of D_{2m+1} is at most $m+1$. To achieve this bound, the geometric approach to the descent algebra is used, in which the descent algebra of a finite Coxeter group W is identified with an algebra associated to the reflection arrangement of W.

1. Introduction

Cédric Bonnafé and Götz Pfeiffer determined the Loewy length of the descent algebra $\Sigma_k(W)$ for irreducible finite Coxeter groups W of all types except D_{2m+1} [Bonnafé and Pfeiffer, 2006]. (The case of the symmetric group was established earlier; see [Garsia and Reutenauer, 1989, Theorems 5.6 and 5.7] and [Atkinson, 1992, Theorem 3.4]). For type D_{2m+1}, Bonnafé and Pfeiffer prove that the Loewy length of $\Sigma_k(D_{2m+1})$ is at least $m+2$, and state that they suspect this is an equality. In this paper, we show that $m+2$ is also an upper bound—by showing that the length of the longest path in the quiver of $\Sigma_k(D_{2m+1})$ is at most $m+1$—confirming their suspicion.

We briefly outline the argument, and the structure of the paper. The plan is to use the *geometric approach* to the descent algebra: the descent algebra $\Sigma_k(W)$ can be identified with a subalgebra of an algebra $k\mathcal{F}$ associated to the reflection arrangement of W. This is explained in Section 2. Specifically, there is an action of W on $k\mathcal{F}$ and an anti-isomorphism between $\Sigma_k(W)$ and the W-invariant subalgebra $(k\mathcal{F})^W$. After defining quivers and path algebras in Section 3, Section 4 describes the quiver $\mathcal{Q}$ of $k\mathcal{F}$ and the construction of a W-equivariant surjection $\varphi : k\mathcal{Q} \to k\mathcal{F}$. This results in a surjection $(k\mathcal{Q})^W \twoheadrightarrow (k\mathcal{F})^W$ that we use to gain information about the quiver $\mathcal{Q}_W$ of $(k\mathcal{F})^W$ in Section 5. We then specialize in Section 6 to the irreducible finite Coxeter group of type D and bound the length of the longest path in the quiver $\mathcal{Q}_{D_{2m+1}}$. The reader familiar with the above theory may decide to begin in Section 6.

Throughout this paper k denotes a field whose characteristic is zero or does not divide the order of the Coxeter group.

2. The Geometric Approach to the Descent Algebra

We begin by recalling the definition of Coxeter systems and the descent algebra. We then explain the connection between the descent algebra and the face semigroup algebra of the reflection arrangement of the Coxeter group.

2.1. **Coxeter systems and reflection arrangements.** Let V be a finite dimensional real vector space. A ***finite Coxeter group*** W is a finite group generated by a set of reflections of V. The ***reflection arrangement*** of W is the hyperplane arrangement $\mathcal{A}$ consisting of the hyperplanes of V fixed by some reflection in W.

Let W denote a finite Coxeter group with reflection arrangement $\mathcal{A}$ and let c denote a connected component of the complement of $\bigcup_{H\in\mathcal{A}} H$ in V. A ***wall*** of c is a hyperplane $H \in \mathcal{A}$ such that $H \cap \overline{c}$ spans H, where $\overline{c}$ is the closure of c in V. Let $S \subseteq W$ denote the set of reflections in the walls of c. Then S is a generating set of W [Brown, 1989, §I.5A] and the pair (W, S) is called a ***Coxeter system*** with ***fundamental chamber*** c.

2.2. **The descent algebra.** Fix a Coxeter system (W, S). For $J \subseteq S$, let $W_J = \langle J\rangle$ denote the subgroup of W generated by the elements in J. Each coset of W_J in W contains a unique element of minimal length, where the ***length*** $\ell(w)$ of an element w of W is the smallest number of generators $s_1, \ldots, s_i \in S$ such that $w = s_1 \cdots s_i$ [Humphreys, 1990, Proposition 1.10(c)]. Let X_J denote the set of ***minimal length coset representatives*** of W_J and let $x_J = \sum_{w\in X_J} w$ denote the formal sum of the elements of X_J. Then x_J is an element of the group algebra kW of W with coefficients in a field k. Louis Solomon proved that the elements x_J, one for each $J \subseteq S$, form a basis of a subalgebra of kW [Solomon, 1976, Theorem 1]. This subalgebra is denoted by $\Sigma_k(W)$ and is called the ***descent algebra*** of W.

2.3. **The faces of $\mathcal{A}$.** For each hyperplane $H \in \mathcal{A}$, let H^+ and H^- denote the two open half spaces of V determined by H. The choice of labels H^+ and H^- is arbitrary, but fixed throughout. For convenience, let $H^0 = H$. A ***face*** of the arrangement $\mathcal{A}$ is a non-empty intersection of the form $x = \bigcap_{H\in\mathcal{A}} H^{\sigma_H(x)}$, where $\sigma_H(x) \in \{+, 0, -\}$ for each hyperplane $H \in \mathcal{A}$. The sequence $\sigma(x) = (\sigma_H(x))_{H\in\mathcal{A}}$ is called the ***sign sequence*** of x. The set $\mathcal{F}$ of all faces of $\mathcal{A}$ is a partially order set with partial order given by $x \le y$ iff $x \subseteq \overline{y}$, where $\overline{y}$ denotes the closure of the set y. A ***chamber*** of $\mathcal{A}$ is a face that is maximal with respect to this order.

2.4. **The intersection lattice.** For each face $x \in \mathcal{F}$, the ***support*** $\operatorname{supp}(x)$ of x is the intersection of all hyperplanes in $\mathcal{A}$ that contain x. Equivalently, $\operatorname{supp}(x)$ is the subspace of V spanned by x. The ***dimension*** of x is the dimension of the subspace $\operatorname{supp}(x)$. The ***intersection lattice*** $\mathcal{L}$ of $\mathcal{A}$ is the image of supp; that is, $\mathcal{L} = \operatorname{supp}(\mathcal{F})$. The elements of $\mathcal{L}$ are subspaces of V and are ordered by inclusion. With this partial order, $\mathcal{L}$ is a finite lattice, where the meet of two subspaces is their intersection, and the join of two subspaces is the smallest subspace that contains both. It follows that $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$ is an order-preserving surjection of posets. (N.B. Some authors order $\mathcal{L}$ by reverse inclusion rather than inclusion.)

2.5. **The face semigroup algebra.** Define the ***product of two faces*** $x, y \in \mathcal{F}$ to be the face xy with sign sequence $(\sigma_H(xy))_{H\in\mathcal{A}}$ given by

$$\sigma_H(xy) = \begin{cases} \sigma_H(x), & \text{if } \sigma_H(x) \neq 0, \\ \sigma_H(y), & \text{if } \sigma_H(x) = 0, \end{cases}$$

where $\sigma(x)$ and $\sigma(y)$ are the sign sequences of x and y. This product has a geometric interpretation: the product xy of two faces x and y is the face entered by moving a small positive distance along a straight line from a point in x to a point in y. It is straightfoward to verify that this product gives $\mathcal{F}$ the structure of an associative semigroup with identity, and that $x^2 = x$ and $xyx = xy$ for all $x, y \in \mathcal{F}$. (A semigroup satisfying these identities is called a *left regular band*.)

The semigroup algebra $k\mathcal{F}$ is called the ***face semigroup algebra*** of $\mathcal{A}$ over the field k. It consists of finite k-linear combinations of elements of $\mathcal{F}$ with multiplication induced by the product defined on elements of $\mathcal{F}$.

2.6. **The invariant subalgebra.** Since W is a group of orthogonal transformations of the vector space V, there is an action of W on V defined by setting $w(\vec{v})$ to be the image of $\vec{v} \in V$ under the transformation w. Under this action the set $\mathcal{A}$ is permuted [Humphreys, 1990, Proposition 1.2], so there is an induced action of W on $\mathcal{L}$ and on $\mathcal{F}$. The action preserves the semigroup structure of $\mathcal{F}$, so it extends linearly to an action on $k\mathcal{F}$. Let $(k\mathcal{F})^W$ denote the subalgebra of $k\mathcal{F}$ consisting of the elements of $k\mathcal{F}$ fixed by all elements of W:

$$(k\mathcal{F})^W = \Big\{ a \in k\mathcal{F} : w(a) = a \text{ for all } w \in W \Big\}.$$

The following was first proved by T. P. Bidigare [Bidigare, 1997]. Another proof was given by K. S. Brown and can be found in [Brown, 2000, Theorem 7] or [Saliola, 2007, Theorem 2.7].

Theorem 2.1. *Let W be a finite reflection group and let $k\mathcal{F}$ denote the face semigroup algebra of the reflection arrangement of W. The W-invariant subalgebra $(k\mathcal{F})^W$ is anti-isomorphic to the descent algebra $\Sigma_k(W)$ of W.*

We briefly describe an anti-isomorphism. The faces of the fundamental chamber c are parametrized by the subsets of S: if $J \subseteq S$, then there is a unique face $\mathfrak{c}_J$ of the fundamental chamber c that is fixed by all elements of J [Brown, 1989, §I.5F]. Moreover, every face of $\mathcal{A}$ is in the W-orbit of a unique face of c [Brown, 1989, §I.5F]. So if $\mathcal{O}_J$ denotes the W-orbit of $\mathfrak{c}_J$, then the elements $\boldsymbol{x}_J = \sum_{y \in \mathcal{O}_J} y$, one for each $J \subseteq S$, form a basis of $(k\mathcal{F})^W$. The function $(k\mathcal{F})^W \to \Sigma_k(W)$ defined by mapping $\boldsymbol{x}_J$ to x_J is an anti-isomorphism.

3. Quivers and Path Algebras

Let k be a field and A a finite dimensional k-algebra.

3.1. **Complete system of primitive orthogonal idempotents.** An element $a \in A$ is an ***idempotent*** if $e^2 = e$. Two idempotents $e, f \in A$ are ***orthogonal*** if $ef = 0 = fe$. An idempotent $e \in A$ is ***primitive*** if it cannot be written as $e = f + g$ with f and g non-zero orthogonal idempotents of A. A ***complete system of primitive orthogonal idempotents*** of A is a set $\{e_1, e_2, \ldots, e_n\}$ of primitive idempotents of A that are pairwise orthogonal and that sum to 1_A.

3.2. **The quiver of a split basic algebra.** The ***Jacobson radical*** $\operatorname{rad} A$ of A is the smallest ideal of A such that $A/\operatorname{rad} A$ is semisimple. If $A/\operatorname{rad} A$ is a direct product of copies of k, then A is a ***split basic k-algebra***. Equivalently, A is a split basic algebra if and only if all the simple A-modules are of dimension one.

The ***quiver*** Q of a split basic finite dimensional k-algebra A is the finite directed graph constructed as follows. Let $\{e_v : v \in \mathcal{I}\}$ be a complete system of primitive

orthogonal idempotents of A, where $\mathcal{I}$ is some index set. The vertex set of Q is the index set $\mathcal{I}$, so there is one vertex v in Q for each idempotent e_v. If $x, y \in \mathcal{I}$, then the number of arrows $x \to y$ is $\dim_k e_y\big(\operatorname{rad}(A)/\operatorname{rad}^2(A)\big)e_x$. This construction does not depend on the complete system of primitive orthogonal idempotents, so Q is canonically defined.

3.3. **The path algebra.** The ***path algebra*** kQ of a quiver Q is the k-algebra with basis the set of paths in Q and with multiplication defined on paths by

$$(w_0 \to \cdots \to w_s) \cdot (v_0 \to \cdots \to v_r) = \begin{cases} (v_0 \to \cdots \to v_r \to w_1 \to \cdots \to w_s), & \text{if } w_0 = v_r, \\ 0, & \text{if } w_0 \neq v_r, \end{cases}$$

where $(w_0 \to \cdots \to w_s)$ and $(v_0 \to \cdots \to v_r)$ are paths in Q. If F denotes the ideal of kQ generated by the arrows of Q, then an ideal $I \subseteq kQ$ is said to be ***admissible*** if $F^m \subseteq I \subseteq F^2$ for some $m \in \mathbb{N}$.

If Q is the quiver of A, then there is a surjection $\varphi : kQ \to A$ defined by mapping each vertex x to the idempotent e_x and by mapping the arrows from x to y to elements in $e_y \operatorname{rad}(A) e_x$ whose image in $e_y\big(\operatorname{rad}(A)/\operatorname{rad}^2(A)\big)e_x$ forms a basis of the quotient space. Moreover, $\ker \varphi$ is an admissible ideal of kQ.

3.4. **Loewy length.** The ***Loewy length*** $\operatorname{LL}(A)$ of a finite dimensional k-algebra A is the smallest $l \in \mathbb{N}$ such $(\operatorname{rad} A)^l = 0$. The following observation is pertinent.

Lemma 3.1. *Suppose Q is a finite acyclic quiver. If $A \cong kQ/I$ for some quiver Q and some admissible ideal I of kQ, then $\operatorname{LL}(A) \leq l + 1$, where l is the length of the longest path in Q.*

Proof. Let $\varphi : kQ \to A$ denote a surjection with kernel I. If $F \subseteq kQ$ denotes the ideal generated by the arrows of Q, then $\varphi(F^l) = (\operatorname{rad} A)^l$ for all $l \geq 1$ [Assem et al., 2006, Corollary 2.11]. So if $l \in \mathbb{N}$ is the length of the longest path in Q, then $F^{l+1} = 0$. Hence, $(\operatorname{rad} A)^{l+1} = 0$. Thus, $\operatorname{LL}(A) \leq l + 1$. □

4. A W-Equivariant Surjection

Notation. Throughout this section: (W, S) is a Coxeter system with fundamental domain c; $\mathcal{A}$ is the reflection arrangement of W; $\mathcal{L}$ is the intersection lattice of $\mathcal{A}$; and $k\mathcal{F}$ is the face semigroup algebra of $\mathcal{A}$, where k is a field whose characteristic does not divide the order of W.

In this section we recall the construction a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$ and the construction of a W-equivariant surjection $\varphi : k\mathcal{Q} \to k\mathcal{F}$, where $\mathcal{Q}$ is the quiver of $k\mathcal{F}$.

4.1. **The orbit poset.** For each $x \in \mathcal{F}$ let $\mathcal{O}_x = \{w(x) : w \in W\}$ denote the W-orbit of x, and for each $X \in \mathcal{L}$ let $\mathcal{O}_X = \{w(X) : w \in W\}$ denote the W-orbit of X. The W-orbits of elements of $\mathcal{L}$ form a poset $\mathcal{L}/W = \{\mathcal{O}_X : X \in \mathcal{L}\}$ with partial order given by $\mathcal{O}_X \leq \mathcal{O}_Y$ if and only if there exists $w \in W$ with $w(X) \leq Y$.

Remark 4.1. The poset $\mathcal{L}/W$ is isomorphic to a poset of equivalence classes of subsets of S. Indeed, define a relation on subsets $J, K \subseteq S$ by setting $J \sim K$ if and only if $\operatorname{supp}(\mathfrak{c}_J)$ and $\operatorname{supp}(\mathfrak{c}_K)$ belong to the same orbit. Equivalently, $J \sim K$ if and only if W_J and W_K are conjugate subgroups of W. The poset $S/\!\sim$, with partial order induced by reverse inclusion of subsets of S, is isomorphic to $\mathcal{L}/W$.

4.2. **Complete system of primitive orthogonal idempotents.** The construction requires, for each $X \in \mathcal{L}$, a linear combination $\ell(X)$ of faces of support X with coefficients summing to 1. Moreover, the elements $\ell(X)$ need to satisfy $w(\ell(X)) = \ell(w(X))$ for all $w \in W$.

We provide one example of such elements; see §3.4 of [Saliola, 2007] for other examples. For every orbit $\mathcal{O} \in \mathcal{L}/W$, fix a face $f_{\mathcal{O}}$ such that $\operatorname{supp}(f_{\mathcal{O}}) \in \mathcal{O}$. For each $X \in \mathcal{L}$, let $f_X = f_{\mathcal{O}_X}$ and define

$$\ell(X) = \frac{1}{\lambda_X} \sum_{\substack{z \in \mathcal{O}_{f_X} \\ \operatorname{supp}(z) = X}} z, \qquad \text{where } \lambda_X = |\{z \in \mathcal{O}_{f_X} : \operatorname{supp}(z) = X\}|. \tag{4.1}$$

Note that λ_X is the index of the stabilizer subgroup $W_x = \{w \in W : w(x) = x\}$ of x in the stabilizer subgroup $W_X = \{w \in W : w(X) = X\}$ of X, where x is any face with support X. Hence, λ_X depends only on the orbit of X and so the elements $\ell(X)$ satisfy $w(\ell(X)) = \ell(w(X))$ for all $w \in W$.

Define elements $e_X \in k\mathcal{F}$, one for each $X \in \mathcal{L}$, recursively by the formula

$$e_X = \ell(X) - \ell(X) \sum_{Y > X} e_Y. \tag{4.2}$$

These elements form a complete system of primitive orthogonal idempotents for $k\mathcal{F}$ [Saliola, 2006, Theorem 5.2]. Moreover, they satisfy $w(e_X) = e_{w(X)}$ for all $w \in W$ and all $X \in \mathcal{L}$, so the elements

$$\varepsilon_{\mathcal{O}} = \sum_{X \in \mathcal{O}} e_X, \tag{4.3}$$

one for each $\mathcal{O} \in \mathcal{L}/W$, form a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$ [Saliola, 2007, Theorem 3.7].

Remark 4.2. The above leads to a construction of a complete system of primitive orthogonal idempotents directly within the descent algebra $\Sigma_k(W)$. Let $S/{\sim}$ denote the poset defined in Remark 4.1. For each $\mathcal{O} \in S/{\sim}$, fix a subset $J_{\mathcal{O}} \subseteq S$ with $J_{\mathcal{O}} \in \mathcal{O}$ and define elements $\varepsilon_{\mathcal{O}}$, one for each $\mathcal{O} \in S/{\sim}$, recursively by the formula

$$\varepsilon_{\mathcal{O}} = \frac{1}{\lambda_{\mathcal{O}}} x_{J_{\mathcal{O}}} - \sum_{\mathcal{O}' > \mathcal{O}} \varepsilon_{\mathcal{O}'} \left(\frac{1}{\lambda_{\mathcal{O}}} x_{J_{\mathcal{O}}} \right),$$

where $\lambda_{\mathcal{O}}$ is the index of W_J in the normalizer of W_J. These elements correspond, under the anti-isomorphism of §2.6, to the elements defined in Equation 4.3 for a suitable choice of $f_{\mathcal{O}}$. Therefore, they form a complete system of primitive orthogonal idempotents for $\Sigma_k(W)$. See [Saliola, 2007, Proposition 3.9] for details.

4.3. **The quiver of $k\mathcal{F}$.** The quiver of $k\mathcal{F}$ is the directed graph $\mathcal{Q}$ constructed as follows. The vertex set of $\mathcal{Q}$ is $\mathcal{L}$, and there is exactly one arrow $X \to Y$, for $X, Y \in \mathcal{L}$, if and only if $Y \lessdot X$. There exists a surjection $\varphi : k\mathcal{Q} \to k\mathcal{F}$ with kernel generated by the sum of all the paths in $\mathcal{Q}$ of length two. A proof of this for any central hyperplane arrangement can by found in [Saliola, 2006]. Below we recall the construction of a W-equivariant $\varphi : k\mathcal{Q} \twoheadrightarrow k\mathcal{F}$. See [Saliola, 2007, §4] for details.

Define an action of W on the path algebra $k\mathcal{Q}$ as follows. Fix an orientation ϵ_X on each subspace $X \in \mathcal{L}$. Thus, ϵ_X is a map that assigns 1 or -1 to a basis of X depending on whether the basis is positively or negatively oriented. For $w \in W$ and

$X \in \mathcal{L}$, let

$$\sigma_X(w) = \epsilon_X\Big(\vec{x}_1, \dots, \vec{x}_s\Big)\epsilon_{w(X)}\Big(w(\vec{x}_1), \dots, w(\vec{x}_s)\Big),$$

where $\vec{x}_1, \dots, \vec{x}_s$ is a basis of the subspace X. Note that if $w(X) = X$, then $\sigma_X(w)$ is 1 if the restriction $w|_X$ of w to X is orientation-preserving, and is -1 otherwise. For $w \in W$ and a path $(X_0 \to \cdots \to X_t)$ in $\mathcal{Q}$ define

$$w\,(X_0 \to \cdots \to X_t) = \sigma_{X_0}(w)\sigma_{X_t}(w)\Big(w(X_0) \to \cdots \to w(X_t)\Big),$$

where $w(X_i)$ is the image of $X_i \in \mathcal{L}$ under the action of W on $\mathcal{L}$. The following is Theorem 4.7 of [Saliola, 2007].

Theorem 4.3. *There exists a W-equivariant k-algebra surjection $\varphi : k\mathcal{Q} \twoheadrightarrow k\mathcal{F}$. That is, φ satisfies $\varphi(w(a)) = w(\varphi(a))$ for all $w \in W$ and all $a \in k\mathcal{Q}$. In addition,* $\ker(\varphi)$ *is generated by the sum of all the paths of length two in $\mathcal{Q}$.*

We briefly recall that construction. Let ϵ_X denote the orientations chosen above. Define φ on each vertex X and arrow $X \to Y$ of $\mathcal{Q}$ by

$$\varphi(X) = e_X \qquad \text{and} \qquad \varphi(X \to Y) = \lambda_Y\, e_Y \left(\sum_{x > y} [y : x] x\right) e_X,$$

where λ_Y is defined in Equation (4.1), where y is any face of support Y and

$$[y : x] = \epsilon_{\operatorname{supp}(y)}(\vec{y}_1, \dots, \vec{y}_t)\epsilon_{\operatorname{supp}(x)}(\vec{y}_1, \dots, \vec{y}_t, \vec{x}_1),$$

where $\vec{y}_1, \dots, \vec{y}_t$ is a basis of $\operatorname{supp}(y)$ and $\vec{x}_1$ is a vector in x. Then φ extends linearly and multiplicatively to a W-equivariant k-algebra surjection $\varphi : k\mathcal{Q} \to k\mathcal{F}$.

5. On the Quiver of $(k\mathcal{F})^W$

We continue with the notation of the previous section and let $\mathcal{Q}_W$ denote the quiver of $(k\mathcal{F})^W$. The quiver $\mathcal{Q}_W$ is not known for arbitrary W, but the idempotents of Equation (4.3) and the surjection of Theorem 4.3 provide some information about the structure of $\mathcal{Q}_W$. In the next section we specialize to W of type D.

The following result is our main tool.

Lemma 5.1. *If for every path P in $\mathcal{Q}$ that begins at a vertex in $\mathcal{O}' \in \mathcal{L}/W$ and ends at a vertex in $\mathcal{O} \in \mathcal{L}/W$ there exists $w \in W$ such that $w(P) = -P$, then there is no arrow from $\mathcal{O}'$ to $\mathcal{O}$ in $\mathcal{Q}_W$.*

Proof. If there is an arrow $\mathcal{O}' \to \mathcal{O}$, then the vector space $\varepsilon_{\mathcal{O}}(k\mathcal{F})^W\varepsilon_{\mathcal{O}'}$ is nonzero (see §3.2). We'll show that this vector space is zero if the hypothesis holds.

For each $\mathcal{O} \in \mathcal{L}/W$, let $\nu_{\mathcal{O}} = \sum_{X \in \mathcal{O}} X \in k\mathcal{Q}$. Let $\varphi : k\mathcal{Q} \to k\mathcal{F}$ denote the W-equivariant surjection of Theorem 4.3. Then φ restricts to a surjection

$$\nu_{\mathcal{O}}(k\mathcal{Q})^W\nu_{\mathcal{O}'} \twoheadrightarrow \varepsilon_{\mathcal{O}}(k\mathcal{F})^W\varepsilon_{\mathcal{O}'}.$$

We'll show that $\nu_{\mathcal{O}}(k\mathcal{Q})^W\nu_{\mathcal{O}'} = 0$. This subspace is spanned by elements of the form $\sum_{P' \in \mathcal{O}_P} P'$, where P is a path of $\mathcal{Q}$ that begins at a vertex in $\mathcal{O}'$ and ends at a vertex in $\mathcal{O}$, and where $\mathcal{O}_P$ is the W-orbit of P. The hypothesis implies $w(P) = -P$ for some $w \in W$, so

$$\sum_{P' \in \mathcal{O}_P} P' = \sum_{P' \in \mathcal{O}_P} w(P') = \sum_{P' \in \mathcal{O}_{-P}} P' = -\sum_{P' \in \mathcal{O}_P} P'.$$

Therefore, $\sum_{P' \in \mathcal{O}_P} P' = 0$. So $\nu_{\mathcal{O}}(k\mathcal{Q})^W\nu_{\mathcal{O}'} = 0$. □

Our first result on the structure of $\mathcal{Q}_W$ shows that it contains no oriented cycles.

Proposition 5.2. *There is exactly one vertex in $\mathcal{Q}_W$ for each element of $\mathcal{L}/W$. If $\mathcal{O}'\to\mathcal{O}$ is an arrow in $\mathcal{Q}_W$, then $\mathcal{O} \leq \mathcal{O}'$ in $\mathcal{L}/W$. In particular, $\mathcal{Q}_W$ does not contain any oriented cycles.*

Proof. Since the elements in Equation (4.3) form a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$, the vertex set of $\mathcal{Q}_W$ is the poset $\mathcal{L}/W$.

If $(X_0\to\cdots\to X_l)$ is a path in $\mathcal{Q}$, then $X_l \leq X_0$. In particular, $\mathcal{O}_{X_l} \leq \mathcal{O}_{X_0}$. So if $\mathcal{O} \not\leq \mathcal{O}'$, then the condition of Lemma 5.1 is satisfied. Therefore, there is no arrow from $\mathcal{O}'$ to $\mathcal{O}$ in $\mathcal{Q}_W$. It follows that $\mathcal{Q}_W$ cannot contain an oriented cycle. □

Our next result shows that the quiver $\mathcal{Q}_W$ contains at least one isolated vertex.

Proposition 5.3. *There are no arrows in $\mathcal{Q}_W$ beginning at $\{V\}$.*

Proof. Let $(X_0\to\cdots\to X_l)$ be a path in $\mathcal{Q}$ with $X_0 = V$. Let $w \in W$ denote the reflection in the hyperplane X_1. Then

$$\begin{aligned} w\,(X_0\to\cdots\to X_l) &= \sigma_{X_0}(w)\sigma_{X_l}(w)\,(w(X_0)\to\cdots\to w(X_l)) \\ &= -\,(X_0\to\cdots\to X_l)\,. \end{aligned}$$

By Lemma 5.1, there is no arrow in $\mathcal{Q}_W$ beginning at $\{V\}$. □

6. The Loewy Length of $\Sigma_k(D_{2m+1})$

Notation. Throughout this section: D_n is a Coxeter group of type D, $\mathcal{A}$ is the reflection arrangement of D_n; $\mathcal{L}$ is the intersection lattice of $\mathcal{A}$; and $k\mathcal{F}$ is the face semigroup algebra of $\mathcal{A}$, where k is a field whose characteristic does not divide the order of D_n.

6.1. **Coxeter groups of type D.** For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$ and let $[\pm n] = \{1, 2, \ldots, n\} \cup \{-1, -2, \ldots, -n\}$. A ***signed permutation*** of $[\pm n]$ is a permutation w of $[\pm n]$ satisfying $w(-i) = -w(i)$ for all $i \in [n]$. A signed permutation w of $[\pm n]$ acts on $\mathbb{R}^n$ by permuting and negating coordinates:

$$w(v_1, v_2, \ldots, v_n) = \left(v_{w^{-1}(1)}, v_{w^{-1}(2)}, \ldots, v_{w^{-1}(n)}\right), \text{ where } v_{-i} = -v_i \text{ for } i \in [n].$$

The Coxeter group D_n is the subgroup of the group of signed permutations of $[\pm n]$ that negate an even number of elements of $[n]$. The reflection arrangement $\mathcal{A}$ of D_n consists of the hyperplanes

$$H_{ij} = \{\vec{v} \in \mathbb{R}^n : v_i = v_j\}, \text{ where } i \neq j, -i \in [\pm n].$$

6.2. **Intersection lattice.** A ***set partition*** of $[\pm n]$ is a collection of nonempty subsets $B = \{B_1, \ldots, B_r\}$ of $[\pm n]$ such that $\bigcup_i B_i = [\pm n]$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. The sets B_i in B are called the ***blocks*** of B. The collection of set partitions of a finite set form a finite lattice, with partial order given by: $B \leq C$ if and only if every block of C is contained in a block of B. If $A \subseteq [\pm n]$, then $\overline{A} = \{-a : a \in A\}$.

The intersection lattice $\mathcal{L}$ of $\mathcal{A}$ is isomorphic to the sublattice of set partitions of $[\pm n]$ of the form $\{B_1, \ldots, B_r, C, \overline{B}_r, \ldots, \overline{B}_1\}$, where C satisfies: $C = \emptyset$ or $|C| \geq 4$; and $C = \overline{C}$ [Barcelo and Ihrig, 1999, Theorem 4.1]. The isomorphism is given by

$$\begin{aligned} &P = \{P_1, \ldots, P_r\} \mapsto \\ &\qquad \left\{\vec{v} \in V : v_i = v_j \text{ if } i, j \in P_h \text{ for some } h \in [r]\right\} = \bigcap_{h=1}^{r} \left(\bigcap_{i,j \in P_h} H_{ij} \right), \end{aligned}$$

where P is a set partition of $[\pm n]$ and $v_{-i} = -v_i$ for $i \in \mathbb{N}$.

To simplify notation, we let $\pi(X)$ denote the set partition of $[\pm n]$ induced by $X \in \mathcal{L}$, and we let $\{B_1, \ldots, B_r; C\}$ denote the set partition $\{B_1, \ldots, B_r, C, \overline{B}_r, \ldots, \overline{B}_1\}$. The block C is called the ***central block***. Under this isomorphism the action of D_n on $X \in \mathcal{L}$ is given by permuting the elements of $\pi(X)$. That is, $\pi(w(X)) = w(\pi(X))$ for all $w \in D_n$ and $X \in \mathcal{L}$.

6.3. **Canonical basis.** The set partition $\pi(X) = \{B_1, \ldots, B_r; C\}$ describes a basis of X. For each $i \in [r]$, let

$$\boldsymbol{\beta}_i = \sum_{j \in B_i} \vec{e}_j,$$

where $\vec{e}_1, \ldots, \vec{e}_n$ is the standard basis of $\mathbb{R}^n$ and $\vec{e}_{-j} = -\vec{e}_j$ for $j \in [n]$. The vectors $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r$ form a basis of the subspace X called the ***canonical basis*** of X.

6.4. **The length of the longest path in $\mathcal{Q}_{D_{2m}}$.** This serves as a quick example to illustrate the approach we take in the following section.

The Coxeter group D_{2m} contains an element w_0 that acts on V by central reflection. That is, $w_0(\vec{v}) = -\vec{v}$ for all $\vec{v} \in V$. Therefore, $\sigma_X(w_0) = (-1)^{\dim(X)}$. So if A is an arrow in $\mathcal{Q}$, then $w_0(A) = -A$. It follows from Lemma 5.1 that there is no arrow $\mathcal{O} \to \mathcal{O}'$ in $\mathcal{Q}_{D_{2m}}$ if $\mathcal{O} \lessdot \mathcal{O}'$. Combined with Proposition 5.3, this implies that the length of the longest path in $\mathcal{Q}_{D_{2m}}$ is at most $\frac{2m-1}{2}$, since $2m$ is the length of the longest path in $\mathcal{Q}$. This establishes the following.

Proposition 6.1. *The length of the longest path in $\mathcal{Q}_{D_{2m}}$ is at most $m - 1$.*

This implies that the Loewy length of the descent algebra $\Sigma_k(D_{2m})$, for $m \geq 2$, is at most m (see the proof Theorem 6.5). Also, the same argument also gives an upper bound of $\left\lceil \frac{n}{2} \right\rceil$ for the Loewy length of the descent algebra $\Sigma_k(B_n)$. These are both equalities [Bonnafé and Pfeiffer, 2006, §5E].

6.5. **The Loewy length of $\Sigma_k(D_{2m+1})$.** In this section we develop necessary conditions on $\mathcal{O}_X$ and $\mathcal{O}_Y$ for there to be an arrow from $\mathcal{O}_X$ to $\mathcal{O}_Y$ in $\mathcal{Q}_{D_{2m+1}}$.

For $X \in \mathcal{L}$, let $\pi(X) = \{B_1, \ldots, B_r; C\}$ denote the set partition induced by X (see §6.2). Let $\mathrm{Even}(X)$ denote the number of $i \in [r]$ with $|B_i|$ even, and let $\mathrm{Odd}(X)$ denote the number of $j \in [r]$ with $|B_j|$ odd.

Lemma 6.2. *If there is an arrow $\mathcal{O}' \to \mathcal{O}$ in $\mathcal{Q}_{D_{2m+1}}$, then* $\mathrm{Even}(Y) \leq \mathrm{Even}(X)$ *for all $X \in \mathcal{O}'$ and $Y \in \mathcal{O}$.*

Proof. We prove that if $\mathrm{Even}(X) < \mathrm{Even}(Y)$, then there is no arrow $\mathcal{O}_X \to \mathcal{O}_Y$.

Suppose P is a path in $\mathcal{Q}$ beginning at a vertex X' in $\mathcal{O}_X$ and ending at a vertex Y' in $\mathcal{O}_Y$. Since $\pi(X')$ and $\pi(X)$ are in the same orbit, $\mathrm{Even}(X') = \mathrm{Even}(X)$. Similarly, $\mathrm{Even}(Y') = \mathrm{Even}(Y)$. So $\mathrm{Even}(X') < \mathrm{Even}(Y')$.

If every even-sized non-central block B_i in $\pi(Y') = \{B_1, \ldots, B_r; C\}$ contains an even-sized non-central block of $\pi(X')$, then $\mathrm{Even}(X') \geq \mathrm{Even}(Y')$, contrary to our assumption. Therefore, for some $i \in [r]$ the block $B_i \in \pi(Y')$ is even-sized and is a union of an even number of odd-sized blocks of $\pi(X')$.

Let w be the signed permutation that negates all elements of B_i. Then $w \in D_{2m+1}$ since $|B_i|$ is even and $w(Z) = Z$ for all vertices of P. Since w negates an even number of blocks of $\pi(X')$ and fixes the others, w negates an even number of vectors (and fixes the others) in a canonical basis of X'. Hence $\sigma_{X'}(w) = 1$. Similarly, w

negates exactly 1 block of $\pi(Y')$ while fixing the others, so $\sigma_{Y'}(w) = -1$. Hence, $w(P) = -P$. By Lemma 5.1 there is no arrow $\mathcal{O}_X \to \mathcal{O}_Y$ in $\mathcal{Q}_{D_{2m+1}}$. □

Lemma 6.3. *If $X' \to Y'$ is an arrow in $\mathcal{Q}$ and if $\mathrm{Odd}(X') \neq 1$, then there is no arrow $\mathcal{O}_{X'} \to \mathcal{O}_{Y'}$ in $\mathcal{Q}_{D_{2m+1}}$.*

Proof. Suppose $X \to Y$ is an arrow in $\mathcal{Q}$ with $X \in \mathcal{O}_{X'}$ and $Y \in \mathcal{O}_{Y'}$. Since $\pi(X')$ is in the orbit of $\pi(X)$, $\mathrm{Odd}(X) = \mathrm{Odd}(X') \neq 1$. Since $Y \lessdot X$, $\pi(Y)$ is obtained from $\pi(X) = \{B_1, \dots, B_r; C\}$ by merging two blocks. Let $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_r$ be the canonical basis for X (see §6.3).

Case 1. $\pi(Y)$ is obtained from $\pi(X)$ by merging B_i and B_j, where $i \neq j$.

If $|B_i \cup B_j|$ is even, then let w be the signed permutation that negates the elements of B_i and B_j. Then w negates two elements of the canonical basis of X and fixes the other basis elements, so $w(X) = X$ and $\sigma_X(w) = 1$. Since $\{\boldsymbol{\beta}_i + \boldsymbol{\beta}_j\} \cup \{\boldsymbol{\beta}_h : h \neq i, j\}$ is a basis of Y, and since w negates $\boldsymbol{\beta}_i + \boldsymbol{\beta}_j$ and fixes the others, it follows that $w(Y) = Y$ and $\sigma_Y(w) = -1$. Therefore, $w(X \to Y) = -(X \to Y)$.

If $|B_i \cup B_j|$ is odd, then let w be the signed permutation that negates the elements of B_i, B_j and the elements of B_h, where $h \neq i, j$ and $|B_h|$ is odd (such a block exists since $\mathrm{Odd}(X) \neq 1$). Then w negates three elements of the canonical basis of X and two elements of the basis $\{\boldsymbol{\beta}_i + \boldsymbol{\beta}_j\} \cup \{\boldsymbol{\beta}_a : a \neq i, j\}$ of Y. Therefore, $\sigma_X(w) = -1$ and $\sigma_Y(w) = 1$. It follows that $w(X \to Y) = -(X \to Y)$.

Case 2. $\pi(Y)$ is obtained from $\pi(X)$ by merging B_i and $\overline{B}_j$, where $i \neq j$.

This is argued as is *Case 1.*

Case 3. $\pi(Y)$ is obtained from $\pi(X)$ by merging the blocks $B_i, \overline{B}_i$ and C.

If $|B_i|$ is even, then let w be the signed permutation that negates the elements of B_i. Then w negates one element of the canonical basis of X and no elements of the basis $\{\boldsymbol{\beta}_a : a \neq i\}$ of Y. Thus, $w(X \to Y) = -(X \to Y)$.

If $|B_i|$ is odd, then let w be the signed permutation that negates the elements of B_i and the elements of B_h, where $h \neq i$ and $|B_h|$ is odd. Then w negates two elements of the canonical basis of X and one element of the basis $\{\boldsymbol{\beta}_a : a \neq i\}$ of Y. It follows that $w(X \to Y) = -(X \to Y)$.

It follows from Lemma 5.1 that there is no arrow $\mathcal{O}_{X'} \to \mathcal{O}_{Y'}$ in $\mathcal{Q}_{D_{2m+1}}$. □

Proposition 6.4. *The length of the longest path in $\mathcal{Q}_{D_{2m+1}}$ is at most $m+1$.*

Proof. Suppose $(\mathcal{O}_0 \to \mathcal{O}_1 \to \cdots \to \mathcal{O}_l)$ is a path in $\mathcal{Q}_{D_{2m+1}}$. Then for $0 \leq i \leq l$, there is $X_i \in \mathcal{O}_i$ such that $X_l \leq \cdots \leq X_1 \leq X_0$. Note that $X_0 \neq V$ by Proposition 5.3.

For each $j \in [l]$, let $d_j = \dim(X_{j-1}) - \dim(X_j)$. If $d_i \geq 2$ for all $i \in [l]$, then

$$2l \leq \sum_{i=1}^{l} d_i = \dim(X_0) - \dim(X_l) \leq \dim(X_0) \leq 2m,$$

so $l \leq m$. Suppose instead that $d_j = 1$ for some $j \in [l]$, and let i be the smallest such j. By the choice of i, $X_{i-1} \to X_i$ is an arrow in $\mathcal{Q}$ with $X_{i-1} \in \mathcal{O}_{i-1}$ and $X_i \in \mathcal{O}_i$. Then $\mathrm{Odd}(X_{i-1}) = 1$ by Lemma 6.3 and $\mathrm{Even}(X_{i-1}) \leq \mathrm{Even}(X_0)$ by Lemma 6.2.

Recall that for each $X \in \mathcal{L}$, if $\pi(X) = \{B_1, \dots, B_r; C\}$, then $\dim(X) = r$ (§6.3). In particular, $\dim(X) = \mathrm{Even}(X) + \mathrm{Odd}(X)$ and $\dim(X) \leq (2m+1) - \mathrm{Even}(X)$. Therefore, since $\mathrm{Even}(X_{i-1}) \leq \mathrm{Even}(X_0)$,

$$\dim(X_0) \leq (2m+1) - \mathrm{Even}(X_0) \leq (2m+1) - \mathrm{Even}(X_{i-1}).$$

By the choice of i, $d_j \geq 2$ for all $j \in [i-1]$, so

$$\begin{aligned} 2(i-1) &\leq \dim(X_0) - \dim(X_{i-1}) \\ &\leq \Big((2m+1) - \mathrm{Even}(X_{i-1})\Big) - \Big(\mathrm{Even}(X_{i-1}) + \mathrm{Odd}(X_{i-1})\Big) \\ &\leq 2\big(m - \mathrm{Even}(X_{i-1})\big). \end{aligned}$$

Since the length of $(\mathcal{O}_{i-1} \rightarrow \cdots \rightarrow \mathcal{O}_l)$ is bounded by $\dim(X_{i-1})$,

$$\begin{aligned} l &= (l-(i-1)) + (i-1) \\ &\leq \dim(X_{i-1}) + \big(m - \mathrm{Even}(X_{i-1})\big) \\ &\leq \big(\mathrm{Even}(X_{i-1}) + \mathrm{Odd}(X_{i-1})\big) + \big(m - \mathrm{Even}(X_{i-1})\big) \\ &\leq m+1. \end{aligned}$$

□

Theorem 6.5. *For all $m \geq 2$, the Loewy length of $\Sigma_k(D_{2m+1})$ is $m+2$.*

Proof. By Theorem 2.1, the Loewy length of $\Sigma_k(D_{2m+1})$ is the Loewy length of $(k\mathcal{F})^{D_{2m+1}}$. By Lemma 3.1 and the previous Proposition, the Loewy length of $(k\mathcal{F})^{D_{2m+1}}$ is bounded by $m+2$. By Corollary 5.9(b) of [Bonnafé and Pfeiffer, 2006], this is also a lower bound. □

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